Presumption and Prejudice in Logical Inference

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ABSTRACT

Two nonstandard modes of inference, confirmation and denial, have been shown by Bandler and Kohout to be valid in fuzzy propositional and predicate logics. If denial is used in combination with modus ponens, the resulting inference mode ("augmented modus ponens") yields more precise bounds on the consequent of an implication than are usually called for in approximate reasoning. Similar results hold for augmented modus tollens constructed from confirmation and conventional fuzzy modus tollens

Two simpler modes of inference, presumption and prejudice, are also valid under the same assumptions as confirmation and denial Prejudice imposes an upper bound on the truth value of the consequent of a fuzzy implication regardless of the truth value of the antecedent; presumption imposes a lower bound on the truth value of the antecedent regardless of that of the consequent. Some of the consequences of presumption and prejudice cast doubt on the suitability of fuzzy propositional and predicate logics for use in expert systems that are designed to process real-world data.

A logic based directly on fuzzy sets is explored as an alternative. Fuzzy set logic supports fuzzy modus ponens and modus tollens but does not entail the more problematic modes of confirmation, denial, presumption, and prejudice However, some of the expressive power derivable from the diversity of fuzzy propositional logics and their derivative fuzzy predicate logics is lost.

KEYWORDS: ***fuzzy logic, inference modes, propositional logic, predicate logic, set logic***

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INTRODUCTION

An elementary logical statement is the basic unit of any logic system, in the sense that it is the smallest logical unit to which a truth value can be assigned within the system One of the most fundamental ways of dividing the general field of logic is by the way statements are analyzed In propositional logic, elementary statements are treated as unanalyzed units; in predicate logic, an elementary statement asserts that a particular object has a particular attribute or that a particular set of objects stand in a particular relation to one another; and in set logic, an elementary statement asserts that an object belongs to a particular set.

In the following section we discuss propositional logic, considering both the traditional binary truth value systems and multivalent fuzzy systems. The latter discussion concentrates on multivalent logical operations that follow the axioms of continuous triangular norms (T-norms) and their dual triangular conorms (Schweizer and Sklar [1, 2]) Drawing upon recent work on the nonstandard inference modes of confirmation and denial (Bandler and Kohout [3], Hall [4], Schwartz [5]), we derive "augmented" versions of modus ponens and modus tollens for multivalent logic, using the method of "residuation" (Trillas and Valverde [6]) to derive both an upper bound and a lower bound for the inferred truth value; these bounds often coincide, yielding a unique answer Two additional modes of inference, presumption and prejudice, are derived using the same techniques; these modes entail a lower bound on the truth of the antecedent and an upper bound on the truth of the consequent from knowledge of the truth of the implication alone

The third section presents a parallel development of similar results for the case of predicate logic It concludes with a simple demonstration of "knowledge base psychosis," in which multiple rules of fuzzy predicate logic interact to preclude the denial of any base value of the antecedent variable due to presumption, and to preclude the affirmation of any base value of the consequent variable due to prejudice

In the fourth section we consider inferences derived from a more fundamental grounding in set logic We consider a single primary universe of discourse consisting of objects that are classified into fuzzy subsets on the basis of the values of several attributes; an inference rule in this sytem consists of the assertion that all objects belong to a consequent fuzzy subset C at least as strongly as they belong to an antecedent fuzzy subset A. In this environment, ordinary fuzzy modus ponens and modus tollens are well-defined according to the "standard strict" implication operator and the compositional rule of inference, but confirmation, denial, presumption, and prejudice are not derivable. Thus, no a priori constraints are placed on the antecedent or the consequent

We next illustrate these results with a simple example based on the fuzzy relation between fast driving and poor fuel economy. We conclude with a summary of the implications of these results and an outline of future research involving the concept of "usuality" in fuzzy set logic

PROPOSITIONAL LOGIC

An implication rule in propositional logic takes the simple form "if ⟨antecedent⟩ then ⟨consequent⟩," or "If A then C" for short In standard two-valued logic, the only way that this implication can be judged false is if A is true while C is false; thus, the implication is held to be true if C is true or if A is false In other words, the truth or falsity of the statement "If A then C" is identical to the truth or falsity of the statement "C or Not A " Many theoretical treatments of material implication in multivalent propositional logic preserve this equivalence; in such a logic the degree of truth attached to "If A then C" can be computed by finding the truth value of "C or Not A " Different systems of multivalent logic arise from the choice of a truth function to represent "or" and to a lesser extent from the representation of "Not " The most important class of such systems is the class of S-implications, in which the "or" operator is defined by a T-conorm (Schweizer and Sklar [1, 2], Bonissone [7]), the most important S-implications are Kleene–Dienes, probabilistic and Lukasiewicz.

Other systems of multivalent logic make a distinction between "If A then C" and "C or Not A," but every propositional implication operator $\rightarrow$ defines a truth value $\mathrm{Tr}(A \rightarrow C) = I[\mathrm{Tr}(A), \mathrm{Tr}(C)]$ that varies directly with $\mathrm{Tr}(C)$ (the truth of C) and inversely with $\mathrm{Tr}(A)$ (the truth of A).[1] The most important class of logics that do not postulate an identity between "If A then C" or "C or Not A" is the class of R-implications, which define the modus ponens operation in terms of a T-norm and derive the implication operator to fit; the most important R-implications are Brouwer (also known as "standard star"), quotient, and Lukasiewicz. (Note that the Lukasiewicz logic satisfies the definitions of both S-implication and R-implication, this versatility is surely not unrelated to the popularity of the logic.)

Several authors have performed more or less empirical comparisons among operators gleaned from the literature, with conflicting results depending on which particular real or simulated domain of application was used (Mizumoto [8, 9], Whalen and Schott [10]). On the theoretical side, other researchers start with a particular set of assumptions and infer an implication operator that suits them (Sanchez [11], Smets and Magrez [12,13]). The safest overall conclusion

[1] The only major exception is Mamdani's operator, which is not a material implication operator ("if–then") but rather a conjunction operator ("and") Inference in Mamdani's system is more properly treated as a generalized alternative syllogism than as generalized modus ponens

from these studies is that different situations appear to require different implication operators.

Modus ponens allows us to infer a lower bound for Tr(C), the truth value of the consequent, from the truth value of the implication Tr($A \rightarrow C$) and the truth value of the antecedent Tr(A); this lower bound is equal to the truth value of "A and (if A then C)," where "and" is defined by the modus ponens generating function mp (Trillas and Valverde [6]) proper to the implication operator I that is used to define "if then"

$$\mathrm{Tr}(C) \geq \mathrm{mp}[\mathrm{Tr}(A), \mathrm{Tr}(A \rightarrow C)]$$

where

$$\mathrm{mp}(x, i) = \inf\{y\colon I(x, y) \geq i\}$$

If the antecedent and the implication are both totally true [Tr(A) = Tr($A \rightarrow C$) = 1], then the lower bound on the truth value of the consequent is 1, or "true," since mp(1, 1) = 1 under any modus ponens generating function mp. One is also the universal upper bound of truth values, so in this case the truth value of the consequent is completely determined as "true" On the other hand, if the antecedent is totally false (truth value = 0), then the lower bound on the truth value of the consequent is only the universal lower bound zero, since mp(0, x) = 0 for any x This does not mean that the consequent is false, only that modus ponens places no restrictions on the truth or falsehood of the consequent For truth values of the antecedent and the implication between total truth and total falsehood, the lower bound for the truth of the consequent varies monotonically

Modus tollens allows us to infer an upper bound for Tr(A), the truth value of the antecedent, from Tr(C) and Tr($A \rightarrow C$) by computing the truth value of "(If A then C) but not C," defining "but not" by the modus tollens generating function mt proper to the implication operator·

$$\mathrm{Tr}(A) \leq \mathrm{mt}[\mathrm{Tr}(C), \mathrm{Tr}(A \rightarrow C)]$$

where

$$\mathrm{mt}(y, i) = \sup\{x\colon I(x, y) \geq i\}$$

If "Not" is defined by any strong negation n such that Tr(Not A) = n[Tr(A)] and Tr(A) = Tr(Not Not A), then the above formulation is equivalent to deriving a lower bound for Tr(Not A) given Tr(Not C) and TR($A \rightarrow C$). For example, if the ubiquitous definition Tr(Not A) = 1 − Tr(A) is used, then

$$\mathrm{Tr}(\mathrm{Not}\ A) \geq 1 - \mathrm{mt}[1 - \mathrm{Tr}(\mathrm{Not}\ C), \mathrm{Tr}(A \rightarrow C)]$$

Several researchers have recently discussed two additional modes of inference in multivalent logic that allow the computation of upper bounds on the truth value of the consequent and lower bounds on the truth value of the antecedent (Bandler and Kohout [3], Hall [4], Schwartz [5]). It is argued that these modes, known as *confirmation* and *denial*, are valid fuzzy or multivalent analog of the invalid classical fallacies of affirming the consequent and denying the antecedent, respectively In denial, the truth values of the implication and antecedent are used in a denial-generating function md to compute an upper bound for the truth value of the consequent:

$$\mathrm{Tr}(C) \leq \mathrm{md}[\mathrm{Tr}(A), \mathrm{Tr}(A \rightarrow C)]$$

where

$$\mathrm{md}(x, i) = \sup\{y: I(x, y) \leq i\}$$

In confirmation, the truth values of the consequent and the implication are used in a confirmation-generating function mc to compute a lower bound for the truth value of the antecedent·

$$\mathrm{Tr}(A) \leq \mathrm{mc}[\mathrm{Tr}(C), \mathrm{Tr}(A \rightarrow C)]$$

where

$$\mathrm{mc}(y, i) = \inf\{x: I(x, y) \leq i\}$$

It is very important to note, however, that confirmation and denial yield no useful results unless the implication is at least partially false; when "If A then C" has a truth value of 1, denial assigns the trivial upper bound of 1 to the truth value of C and confirmation assigns the trivial lower bound of 0 to the truth value of A.[2] (See Hall [4] and Schwartz [5].) The deductive power of both confirmation and denial decreases monotonically as the truth value of the implication increases, in the sense that the bounds they impose on the consequent and the antecedent, respectively, become less restrictive the closer the implication comes to being strictly true. With respect to the other argument, confirmation most strongly constrains the antecedent when the truth value of the consequent is high, while denial most strongly constrains the consequent when the truth value of the antecedent is low.

Given the existence of both an upper and a lower bound on the truth value of the consequent, we define *augmented modus ponens* as an interval-valued function whose result gives both an upper and lower bound on the truth value of the consequent as a joint function of the truth values of the antecedent and of the implication itself.

$$\inf\{y: I(\mathrm{Tr}(A), y) \geq \mathrm{Tr}(A \rightarrow C)\} \leq \mathrm{Tr}(C)$$
$$\leq \sup\{y: I(\mathrm{Tr}(A), y) \leq \mathrm{Tr}(A \rightarrow C)\}$$

or, more concisely,

$$\mathrm{mp}[\mathrm{Tr}(A), \mathrm{Tr}(A \rightarrow C)] \leq \mathrm{Tr}(C) \leq \mathrm{md}[\mathrm{Tr}(A), \mathrm{Tr}(A \rightarrow C)]$$

[2] This is the reason that confirmation and denial allow no valid inferences in classical logic

Modus tollens and confirmation can be unified into *augmented modus tollens* in an exactly similar fashion

$$\inf\{x\colon I[x, \mathrm{Tr}(C)] \le \mathrm{Tr}(A \to C)\} \le \mathrm{Tr}(A) \le \sup\{x\colon I[x, \mathrm{Tr}(C)] \ge \mathrm{Tr}(A \to C)\}$$

or

$$\mathrm{mc}[\mathrm{Tr}(C), \mathrm{Tr}(A \to C)] \le \mathrm{Tr}(A) \le \mathrm{mt}[\mathrm{Tr}(C), \mathrm{Tr}(A \to C)]$$

When $\mathrm{Tr}(A \to C) = 1$, these intervals give no more information than the traditional bounds provided by modus ponens and modus tollens. When $\mathrm{TR}(A \to C)$ is strictly less than 1, augmented modus ponens yields a unique value for $\mathrm{Tr}(C)$ in the logic systems based on the R-implications R_1 (Lukasiewicz), R_2 (quotient), and R_3 (Brouwer) and in the logic systems based on the S-implications I_1 (Lukasiewicz) and I_2 (probabilistic), since the lower bound on $\mathrm{Tr}(C)$ given by $\mathrm{mp}(a, \iota)$ is equal to the upper bound given by $\mathrm{md}(a, \iota)$ for all $\iota < 1$ under these logics. The inferred interval for the S-implication I_3 (Kleene-Dienes) also contains only a single point when $\mathrm{Tr}(A \to C) < 1$ except in the special case where $\mathrm{Tr}(A)$ and $\mathrm{TR}(A \to C)$ are equal. Similarly, augmented modus tollens yields a unique value for $\mathrm{Tr}(A)$ when $\mathrm{TR}(A \to C) < 1$ in the systems R_1, R_2, I_1, and I_2, since $\mathrm{mc}(y, \iota) = \mathrm{mt}(y, \iota)$. For I_3 the value is again unique except when $\mathrm{Tr}(A) = \mathrm{Tr}(A \to C)$, while for R_3 augmented modus tollens always yields a nondegenerate interval of possible truth values of the antecedent A.

The modes of confirmation and denial are closely related to two simpler modes, which we may call the mode of *presumption* and the mode of *prejudice*. In these inference modes, the truth value of the implication alone places limits on the respective truth values of the antecedent and the consequent. According to the mode of prejudice, the truth value of the consequent is always less than or equal to the truth value of the implication According to the mode of presumption, the truth of the antecedent is at least as great as a function of the truth value of the implication, the form of this function varies depending on which implication operator is used $\mathrm{Tr}(A) \ge 1 - \mathrm{Tr}(A \to C)$ under the logic systems I_1 (Lukasiewicz), I_2 (probabilistic), and I_3 (Kleene–Dienes), $\mathrm{Tr}(A) \ge \mathrm{Tr}(A \to C)$ when $\mathrm{Tr}(A \to C) < 1$ under the logic system R_3 (Brouwer·), and $\mathrm{Tr}(A) > 0$ when $\mathrm{Tr}(A \to C) < 1$ under the logic system R_2 (quotient).

Presumption and prejudice can be observed even in the classical system of two-valued logic. When the truth value of an implication is zero in two-valued logic, then "C or Not A" is a completely false statement This implies that "Not (C or Not A)" is a completely true statement "Not (C or Not A)" is equivalent to "A and Not C," so the crisp denial of "If A then C" ensures that A is crisply true (presumption) and that C is crisply false (prejudice). On the other hand, when the truth value of the implication is 1, the upper bound for C is

1 and the lower bound for A is 0, yielding no information at all. In multivalent logic, when the truth value of the implication is neither zero nor one, presumption gives a nontrivial lower bound for the truth value of A and prejudice gives a nontrivial upper bound for the truth value of C

In the case of the S-implications, presumption and prejudice can be easily verified from the equivalence between "If A then C" and "C or Not A," since Tr(C) $\leq$ TR(C or Not A) (prejudice), and the fact that Tr(Not A) $\leq$ Tr(C or Not A) implies that Tr(A) $\geq$ Tr[Not (C or Not A)] (presumption)

In the case of R-implications, prejudice follows directly from the fact that the modus ponens generating function of any R implication is a T-norm one of whose arguments is the truth value of the implication, since the value computed from a T-norm can never exceed either of its arguments Presumption is more difficult with R implications, and in some cases (such as the quotient logic R_2) yields only the nearly trivial constraint that the truth value of the antecedent is strictly greater than zero when the truth value of the implication is not equal to one

For logic systems whose implication operators are neither R-implications nor S-implications, such as early Zadeh or quantum logic (Trillas and Valverde [6]), it is necessary to verify separately each of the six inference modes of modus ponens, modus tollens, confirmation, denial, presumption, and prejudice

While denial requires information about both the truth of A and the truth of the implication in order to specify the least upper bound on the truth of C, it often yields little or no information about C that could not have been derived from the truth of the implication alone using presumption Unless the truth of A is close to the lower bound given by presumption, the upper bound for the truth of C given by denial is not substantially different from that given by prejudice for most common implication operators For Lukasiewicz logic (I_1, R_1), denial varies linearly with Tr(A); for quotient logic (R_2), denial always varies more slowly than Tr(A), for probabilistic logic (I_2), denial varies more slowly than Tr(A) when the latter is close to one but more rapidly when it is close to zero In the Brouwer logic system (R_3) and the Kleene-Dienes logic system (I_3), the bound given by denial is everywhere identical to the bound given by presumption Similarly, the lower bounds on the truth of A provided by confirmation and by presumption are only slightly different in many cases and everywhere identical for R_3 and I_3

Table 1 summarizes the six modes of inference for the logic systems stemming from each of the five most important R and S implication operators

The fact that the inference modes of presumption and prejudice are syntactically valid leads to unfortunate consequences when multivalent propositional logic is used for knowledge-based systems Ordinarily when we give a truth value to an implication rule, we want higher truth values to generate stronger rules and lower truth values to generate weaker ones. However, lowering the truth value of an implication rule makes it more powerful with

Table 1 Inference Modes for S-Implications and R-Implications

Conorm $S(x, y)$	S-Implications $I(a, c) = S(1-a, c)$	Case	Modus ponens $\leq c \leq$	denial	$\leq$ prejudice	Case	Presumption $\leq$	Confirmation	$\leq a \leq$	modus tollens
$S_1 = \mathrm{Min}(1, x+y)$	Lukasiewicz	$1-a<i<1$	$i-(1-a)=c=$	$i-(1-a)$	$\leq i$	$c<i<1$	$1-i\leq$	$1-i+c$	$=a=$	$1-i+c$
	$I_1 = \mathrm{Min}(1, 1-a+c)$	$1-a=i<1$	$0=c=$	0	$\leq i$	$c=i<1$	$1-i\leq$	$1-i+c$	$=a=$	$1-i+c$
		$1-a<i=1$	$a\leq c\leq$	1	$=i$	$c<i=1$	$0=$	0	$\leq a\leq$	c
		$1-a=i=1$	$0\leq c\leq$	1	$=i$	$c=i=1$	$0=$	0	$\leq a\leq$	1
$S_2 = x+y-xy$	Probabilistic	$1-a<i<1$	$[i-(1-a)]/a=c=$	$[i-(1-a)]/a$	$\leq i$	$c<i<1$	$1-i\leq$	$(1-i)/(1-c)$	$=a=$	$(1-i)/(1-c)$
	$I_2 = 1-a+ac$	$1-a=i<1$	$0=c=$	0	$\leq i$	$c=i<1$	$1-i\leq$	1	$=a=$	1
		$1-a<i=1$	$1=c=$	1	$=i$	$c<i=1$	$0=$	0	$=a=$	0
		$1-a=i=1$	$0\leq c\leq$	1	$=i$	$c=i=1$	$0=$	0	$\leq a\leq$	1
$S_3 = \mathrm{Max}(x, y)$	Kleene–Dienes	$1-a<i<1$	$i=c=$	i	$=i$	$c<i<1$	$1-i=$	$1-i$	$=a=$	$1-i$
	$I_3 = \mathrm{Max}(1-a, c)$	$1-a=i<1$	$0\leq c\leq$	i	$=i$	$c=i<1$	$1-i=$	$1-i$	$\leq a\leq$	1
		$1-a<i=1$	$1=c=$	1	$=i$	$c<i=1$	$0=$	0	$=a=$	0
		$1-a=i=1$	$0\leq c\leq$	1	$=i$	$c=i=1$	$0=$	0	$\leq a\leq$	1
Norm $T(x, y)$	R-Implications $R(a, c)$	Case	Modus ponens $\leq c \leq$	denial	$\leq$ prejudice	Case	Presumption $\leq$	Confirmation	$\leq a \leq$	modus tollens
$T_1 =$ $\mathrm{Max}(0, x+y-1)$	Lukasiewicz $R_1 = 1$ if $a\leq c$ else $1-a+c$	$i<1$	$a+i-1=c=$	$i-(1-a)$	$\leq i$	$i<1$	$1-i\leq$	$1-i+c$	$=a=$	$1-i+c$
		$i=1$	$a\leq c\leq$	1	$=i$	$i=1$	$0=$	0	$\leq a\leq$	c
$T_2 = xy$	Quotient $R_2 = 1$ if $a\leq c$ else c/a	$i<1$	$ai=c=$	ai	$\leq i$	$i=0$	$0=$	c	$<a\leq$	1
		$i=1$	$a\leq c\leq$	1	$=i$	$0<i<1$	$0<$	c/i	$=a=$	c/i
						$i=1$	$0=$	0	$\leq a\leq$	c
$T_3 = \mathrm{Min}(x, y)$	Brouwer $R_3 = 1$ if $a\leq c$ else c	$i<1$	$i=c=$	i	$=i$	$i<1$	$i=$	c	$<a<$	1
		$i=1$	$a\leq c\leq$	1	$=i$	$i=1$	$0<$	0	$\leq a\leq$	c

Note $a = \mathrm{Tr}(A)$, $c = \mathrm{Tr}(C)$, $i = \mathrm{Tr}(A \to C)$

resepct to presumption, prejudice, confirmation, and denial while it makes the rule weaker with respect to modus ponens and modus tollens. Even worse, presumption means that implications with low truth values generally impose unwanted restrictions on the input data

However, propositional logic is primarily concerned with absolute statements expressed as unique propositions, so perhaps it is not surprising that a propositional knowledge base handles input data coming from an application environment poorly. Thus, we now turn our attention to predicate logic, which is currently the most widely used logic for knowledge-based systems

PREDICATE LOGIC

The basic form of an implication rule under predicate logic is "If X is A then Y is C," or "If $A(X)$ then $C(Y)$" for short. X and Y are variables with their respective universes of discourse, and A and C are predicates that constrain the values of X and Y. The meaning is that if the predicate A is true of the quantity X, then the predicate C is also true of the quantity Y Our focus will be on implication rules for approximate reasoning in which A and C are fuzzy predicates that impose elastic constraints on X and Y by restricting them to fuzzy subsets of their respective universes of discourse. If the variable X takes on a particular crisp value x_i, then the truth of the predicate A, Tr[$(A(x_i)$], is equal to the membership grade of the x_i in the fuzzy set corresponding to A, $\mu_A(x_i)$, and similarly for Y. Thus, for some values of X and Y the respective predicates A and C will have truth values intermediate between 1 ("true") and 0 ("false") As in propositional logic, the truth value of "If $A(X)$ then $C(Y)$" increases monotonically with the truth value of $C(Y)$ and decreases monotonically with the truth value of $A(X)$.

Given any truth function $I[\mathrm{Tr}(A(X)), \mathrm{Tr}(C(Y))]$ representing implication, it is easy to compute a fuzzy subset of the Cartesian product of the universes of discourse of X and Y, $\mu_{A\to C}(x_i, y_j)$, based on the membership grades of each element x_i in A and y_j in C, $\mu_A(x_i)$ and $\mu_C(y_j)$

$$\mu_{A\to C}(x_1, y_j) = I[\mu_A(x_1), \mu_C(y_j)]$$

Regardless of which implication operator is used, some of the difficulties inherent in approximate reasoning with propositional logic recur in predicate logic when we try to interpret the meaning of the individual membership grade $\mu_{A\to C}(x_i, y_j)$ of an (x_i, y_j) pair in the implication relation. Most fundamentally this membership grade is the truth value of the implication "If x_i belongs to the hypothetical value A, then y_j belongs to the hypothetical value C " Interpreted as such, all of the modes of inference discussed for propositional logic are perfectly valid: $\mu_{A\to C}(x_i, y_j)$ by itself gives a correct lower bound on $\mu_A(x_i)$ through presumption and a correct upper bound on $\mu_C(y_j)$ through prejudice,

$\mu_{A\to C}(x_i, y_j)$ and $\mu_A(x_i)$ together give correct bounds for $\mu_C(y_j)$ through modus ponens and denial; and $\mu_{A\to C}(x_i, y_j)$ and $\mu_C(y_j)$ together give correct bounds for $\mu_A(x_i)$ through modus tollens and confirmation, where by "correct bounds" we simply mean that the actual truth value of the fuzzy predicate in question for any specific x_i or y_j is within the range specified by the inference.

Although this interpretation allows sound inference with respect to the hypothetical values A and C, such inferences are of little or no practical benefit Applied approximate reasoning requires that the knowledge contained in the implication rules be applicable to data arising outside the system To do this, we take $\mu_{A\to C}(x_i, y_j)$ to be the truth value of the implication "If X is x_i then Y is y_j"

In the typical case, generalized modus ponens is then performed using external data about the perceived value of X to make inferences about the unknown value of Y The inferred possibility of each y_j is found in two stages The first stage is to use each x_i's membership grade in the external datum "X is A'" as a truth value for the crisp proposition "X is x_i" in order to perform generalized modus ponens, yielding one lower bound for the truth of "Y is y_j" per base value in the X universe of discourse·

$$\mathrm{Tr}[\text{``}Y = y_j\text{''} \mid A \to C,\ \mathrm{Tr}(\text{``}X = x_i\text{''}) = \mu_{A'}(x)] \geq \mathrm{mp}[\mu_{A'}(x_i),\ \mu_{A\to C}(x_i, y_j)]$$

The second stage is to combine these lower bounds into an overall truth value for "Y is y_j." This is generally done using the max operator S_3, although it may be more appropriate to use the conorm proper to the logic system in use (when this system is not R_3 or I_3). Taken together, the truth values for each individual base value y_j, reinterpreted as membership grades, determine C', the inferred predicate on the Y universe of discourse

However, this procedure leads to a paradox, because if $\mu_{A\to C}(x_i, y_j)$ is less than 1, presumption imposes a priori restrictions on what truth values for "X is x_i" may be asserted, while prejudice imposes a priori restrictions on what truth values for "Y is y_j" may be inferred. And these restrictions on truth values carry over into a priori restrictions on membership grades in A' and C', respectively. In a multirule system, the presumptions and prejudices imposed by the various rules may aggregate to form a "knowledge base psychosis" in which no X value can be denied without violating the presumption of some rule, and no Y value can be affirmed without violating some rule's prejudice[1]

Table 2 shows a simple example of knowledge base psychosis using the Lukasiewicz system with just two simple rules "If X is above medium then Y is high" and "If X is below medium then Y is low." The prejudices of these two rules taken together deny all 11 base values of the Y variable. If the presumptions of the two individual rules are combined, they require the affirmation of nearly all base values of the X variable; furthermore, when presumption is computed from the combined rule derived by the methods of Dubois and Prade [14, 15], it is necessary to affirm all the base values of X.

Formally, any set of axioms that asserts these two rules and also asserts any predicate other than "undefined" for Y or any predicate other than "unknown" for X is logically inconsistent

For a multivalent logic system to be simultaneously useful and logically grounded, a way must be found to preserve modus ponens with respect to external data without simultaneously subjecting that data to presumption and prejudice One promising approach, which we now take up, is a return to the foundations of approximate reasoning, the theory of fuzzy subsets (Zadeh [16])

SET LOGIC

In set logic, the concept of implication is replaced with the more fundamental concept of set inclusion; the set logic relation corresponding to "If A then C" is "All A's are C's," or "If p is an A then p is a C." The latter representation is formally similar to the propositional implication "If X is A then Y is C" except that both the antecedent and the consequent in the set logic form refer to a common universe of discourse Depending on the context, the elements of this universe may be referred to as objects, cases, possible worlds, or some term more specific to the application.

The correspondence between set logic and propositional logic can be further enhanced by using the object–attribute–value formalism. If we define the set A in "If p is an A then p is a C" as the set of object whose X attribute is in the set of X values A and define the set C as the set of objects whose Y attribute is in the set of Y values C, then the antecedent and consequent of the set logic version express exactly the same information as the antecedent and consequent, respectively, of the propositional logic version discussed above. In this case, the rule takes the form "If Xp is in A then Yp is in C," where Xp and Yp are the values of the two attributes X and Y for a single object p—for example, the height and weight of a particular person

Despite the equivalence in expressive power between the two logical systems, the natural rules of inference differ strongly when the sets or equivalent propositions involved are fuzzy ones The most straightforward interpretation of "If p is an A then p is a C" is that the membership grade of any object p in C is at least as great as its membership grade in A:

$$\mu_A(p) \le \mu_C(p)$$

Defining A and C as before in terms of attributes X and Y, we have

$$\mu_A(p) = \mu_A(Xp) \qquad \text{and} \qquad \mu_C(p) = \mu_C(Yp)$$

Then we can state the set inclusion, or implication, relation in terms of these values:

$$\mu_A(Xp) \le \mu_C(Yp) \qquad \text{for all } p$$

Table 2: Knowledge Base Psychosis

	Lukasiewicz min(1 − a + b,1)					If X is below medium then Y is high						
	0	0	0	0	0	0	0	0 25	0 5	0 75	1	Presumption
1	0	0	0	0	0	0	0	0 25	0 5	0 75	1	1
1	0	0	0	0	0	0	0	0 25	0 5	0 75	1	1
1	0	0	0	0	0	0	0	0 25	0 5	0 75	1	1
1	0	0	0	0	0	0	0	0 25	0 5	0 75	1	1
0 75	0 25	0 25	0 25	0 25	0 25	0 25	0 25	0 5	0 75	1	1	0 75
0 5	0 5	0 5	0 5	0 5	0 5	0 5	0 5	0 75	1	1	1	0 5
0 25	0 75	0 75	0 75	0 75	0 75	0 75	0 75	1	1	1	1	0 25
0	1	1	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	1	1	1	0
Prejudice												
	0	0	0	0	0	0	0	0 25	0 5	0 75	1	

	Lukasiewicz min(1 − a + b, 1)					If X is above medium then Y is low						Presumption[a]	
	1	0 75	0 5	0 25	0	0	0	0	0	0	0	Sing	Double
0	1	1	1	1	1	1	1	1	1	1	1	0	1
0	1	1	1	1	1	1	1	1	1	1	1	0	1
0	1	1	1	1	1	1	1	1	1	1	1	0	1
0	1	1	1	1	1	1	1	1	1	1	1	0	1
0 25	1	1	1	1	0 75	0 75	0 75	0 75	0 75	0 75	0 75	0 25	0 75
0 5	1	1	1	0 75	0 5	0 5	0 5	0 5	0 5	0 5	0 5	0 5	0 5

0 75	1	1	0 75	0 5	0 25	0 25	0 25	0 25	0 25	0 25	0 25	0 75	0 75
1	1	0 75	0 5	0 25	0	0	0	0	0	0	0	1	1
1	1	0 75	0 5	0 25	0	0	0	0	0	0	0	1	1
1	1	0 75	0 5	0 25	0	0	0	0	0	0	0	1	1
1	1	0 75	0 5	0 25	0	0	0	0	0	0	0	1	1
Prejudice													
	1	0 75	0 5	0 25	0	0	0	0	0	0	0		
Double prejudice													
	0	0	0	0	0	0	0	0	0	0	0		

	Dubois–Prade combination											Presumption
	0	0	0	0	0	0	0	0 25	0 5	0 75	1	1
	0	0	0	0	0	0	0	0 25	0 5	0 75	1	1
	0	0	0	0	0	0	0	0 25	0 5	0 75	1	1
	0	0	0	0	0	0	0	0 25	0 5	0 75	1	1
	0 25	0 25	0 25	0 25	0	0	0	0 25	0 5	0 75	0 75	1
	0 5	0 5	0 5	0 25	0	0	0	0 25	0 5	0 5	0 5	1
	0 75	0 75	0 5	0 25	0	0	0	0 25	0 25	0 25	0 25	1
	1	0 75	0 5	0 25	0	0	0	0	0	0	0	1
	1	0 75	0 5	0 25	0	0	0	0	0	0	0	1
	1	0 75	0 5	0 25	0	0	0	0	0	0	0	1
	1	0 75	0 5	0 25	0	0	0	0	0	0	0	1
Prejudice												
	0	0	0	0	0	0	0	0	0	0	0	

[a] Double prejudice = minimum prejudice of the two rules, double presumption = maximum presumption of the two rules

If we know the precise value of Xp, we can use modus ponens to make an inference about the value of Yp for the same object The first step is to find the degree to which Xp belongs to A; denote that membership grade as α Note that α is also the membership grade of object p in A, and by the implication relation we can infer that the membership grade of p in C is greater than or equal to α; in other words, p is an element of the α-cut C_α, defined as the set of objects whose membership in C is $\leq \alpha$. This in turn is equivalent to saying that p's value on attribute Y, Yp, belongs to the α-cut C_α, the set of Y values that belong to C at least to degree α. Since α is here a crisp value, C_α is a crisp subset of the Y universe of discourse. In many applications A and C will be convex fuzzy sets, so a crisp value for Xp causes us to infer a crisp interval of values for Yp

If the value of Xp is given as a fuzzy subset A' of the universe of X values, the principle is the same although the procedure is more complicated $\mu_A(Xp)$ becomes $\mu_A(A')$, the degree to which the fuzzy set A' belongs to the fuzzy set A. This membership grade is a fuzzy subset of the universe of ordinary membership grades, given by the following formula

$$\mu_A(A') = \sum_x \frac{\mu_{A'}(x)}{\mu_A(x)}$$

This is also the membership grade of the object p in the fuzzy set of objects A

$$\mu_A(p) = \sum_x \frac{\mu_{A'}(x)}{\mu_A(x)}$$

which by the implication rule gives a fuzzy lower bound on the grade of membership of p in C:

$$\mu_C(p) \geq \sum_x \frac{\mu_{A'}(x)}{\mu_A(x)}$$

and hence a fuzzy lower bound on the grade of membership of Yp in the fuzzy subset C of the universe of Y values

$$\mu_C(Yp) \geq \sum_x \frac{\mu_{A'}(x)}{\mu_A(x)}$$

Taking into account the fact that two or more x values in the X universe of discourse may have the same membership grade in A, we can rewrite this as

$$\mu_C(Yp) \geq \sum_\alpha \frac{\sup_x\{\mu_{A'}(x)\colon \mu_A(x) = \alpha\}}{\alpha}$$

Each element α of this fuzzy set of membership grades corresponds to an α-cut $C_\alpha = \{y\colon \mu_C(y) \geq \alpha\}$ derived from the fuzzy set C given in the rule. Using the resolution identity of fuzzy mathematics (Zadeh [17, 18]), we can combine these α-cuts according to the membership grades of their corresponding α values to form the fuzzy set C', which is our best inference of the value of Yp:

$$C' = \bigcup_{\alpha} \sup_x \{\mu_{A'}(x):\ \mu_A(x) = \alpha\} C_\alpha$$

C' is thus the fuzzy set defined by the fuzzy set of α-cuts of C, in other words, by an α-cut of C for which α is itself a fuzzy set of membership grades

In the case of a crisp data value for Xp, $Xp = x_i$, it is easy to show that this version of modus ponens in fuzzy set logic is equivalent to the use of the standard strict implication operator in fuzzy propostional modus ponens; in both cases, the grade of membership of a particular y_j in C' is 1 if $\mu_A(x_i) \leq \mu_C(y_j)$ and 0 otherwise. The two versions of modus ponens are also equivalent in the case of fuzzy data, the proof, which is omitted here, centers on expressing the inferred C' of propositional modus ponens in terms of its α-cuts using the resolution principle

Despite this equivalence, the problems posed by the modes of presumption and prejudice do not arise If a particular (x_i, y_j) pair has a membership grade of zero in the fuzzy set implication (inclusion) relation, this asserts only that there exists no p such that $Xp = x_i$ and $Yp = y_j$. Without additional information about Yp this leaves Xp unconstrained, and without information about Xp, Yp is similarly unconstrained The fact that the equivalent operation to modus ponens results in a specific fuzzy set of possible values of Yp rather than a lower bound on memberships in this set makes consideration of denial unnecessary

Fuzzy set logic obeys the law of contrapositive symmetry ($A \rightarrow C$ is equivalent to Not $C \rightarrow$ Not A) since $\mu_A(p) \leq \mu_C(p)$ implies that $\mu_{\mathrm{Not}C}(p) \leq \mu_{\mathrm{Not}A}(p)$ Because of this, all of the results given above for modus ponens (and for prejudice and denial) are also true, *mutatis mutandis*, for modus tollens (and for presumption and confirmation)

To summarize, fuzzy set logic succeeds in preserving generalization of the classically valid forms modus ponens and modus tollens, while eliminating the aberrant forms confirmation, denial, presumption, and prejudice. But by mandating the standard strict implication operator, it incurs the cost of requiring a degree of crispness in fuzzy implication that, while convenient for some applications, is quite inappropriate in others.

EXAMPLE: FAST DRIVING IMPLIES POOR FUEL ECONOMY

Propositional Logic

Implication in propositional logic connects two unanalyzed sentences or propositions. Suppose we have the two propositions "You drive fast" or "Your fuel economy is poor'' and the implication ''If you drive fast then your fuel economy is poor.'' If we take the implication to be completely true, this tells us nothing a priori about the individual truth values of ''You drive fast'' and ''Your fuel economy is poor''; but, given any specific truth value x for ''You drive fast,'' modus ponens guarantees that the truth value of ''Your fuel economy is poor'' is between x and 1, regardless of the logic system.

However, suppose we attempt to express reservations about the implication by assigning a reduced truth value, such as .7, to the rule ''If you drive fast then your fuel economy is poor.'' Now, for any of the R and S implications discussed, prejudice demands that the truth value of ''your fuel economy is poor'' must not exceed .7, while presumption demands that the truth value of ''You drive fast'' must be greater than or equal to .3 (R_1, S_2, or S_3), greater than or equal to .7 (R_3), or strictly greater than zero (R_2). Given a truth value for ''You drive fast'' that satisfies presumption, we can use augmented modus ponens to gain information about the truth value of ''Your fuel economy is poor.'' For example, if the truth value of ''You drive fast'' is 8, then the truth value of ''your fuel economy is poor'' is

$$\begin{aligned} .7-(1-.8) &= .5 \text{ according to } R_1 \\ (.7)(.8) &= .56 \text{ according to } R_2 \\ .5/.8 &= .625 \text{ according to } S_2 \\ \text{and} &= .8 \text{ according to } R_3 \text{ or } S_3 \end{aligned}$$

All of these values are precise, in the sense that the lower bound provided by regular modus ponens is equal to the upper bound provided by denial No other truth values for ''Your fuel economy is poor'' are compatible with a truth value of .8 for ''You drive fast'' and a truth value of .7 for ''If you drive fast then your fuel economy is poor'' within the respective logic systems.

Predicate Logic

Fuzzy predicate logic allows us to express the concept of a fast speed as a fuzzy predicate defined over the base variable speed measured in miles per hour (mph) The fuzzy predicate ''Fast'' is induced by a fuzzy subset of speeds as shown in Figure 1 A crisp predicate has the value true (1.0) for some elements of its universe of discourse and false (zero) for others. By extension, a fuzzy predicate has a truth value in the range from zero to one for each element of its universe; thus, for example, the predicate ''Fast'' has the truth value of .8 when the speed is 60 mph. The fuzzy predicate ''Poor,'' whose universe of discourse is fuel economy measured in miles per gallon (mpg), is similarly induced by the fuzzy set shown in Figure 2. For example, the fuzzy predicate ''Poor'' has a truth value of .8 when fuel economy is 19 mpg.

The implication ''If speed is fast then fuel economy is poor'' is also a fuzzy predicate, whose domain of definition is the Cartesian product of mph and mpg.

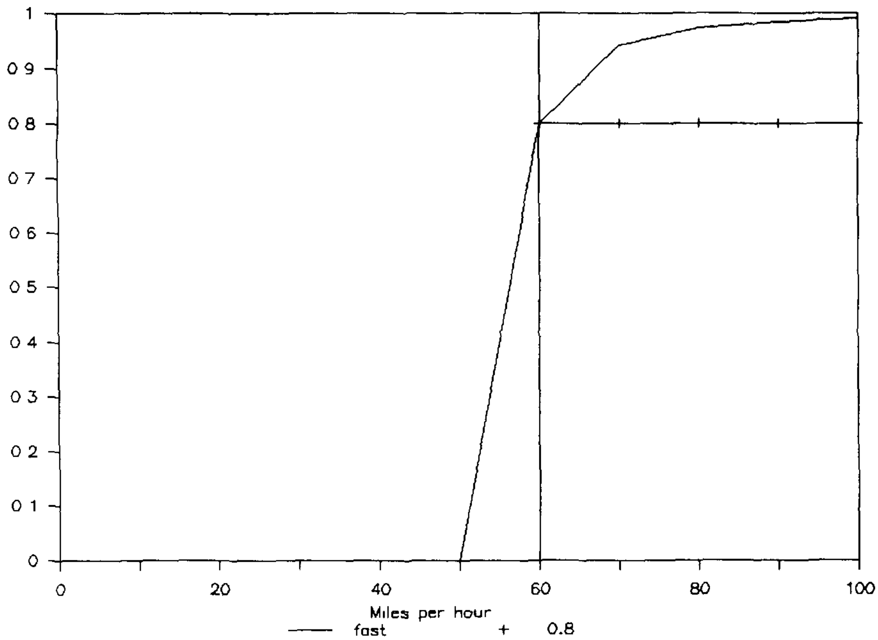

Figure 1. The Fuzzy Set of "Fast" Speeds

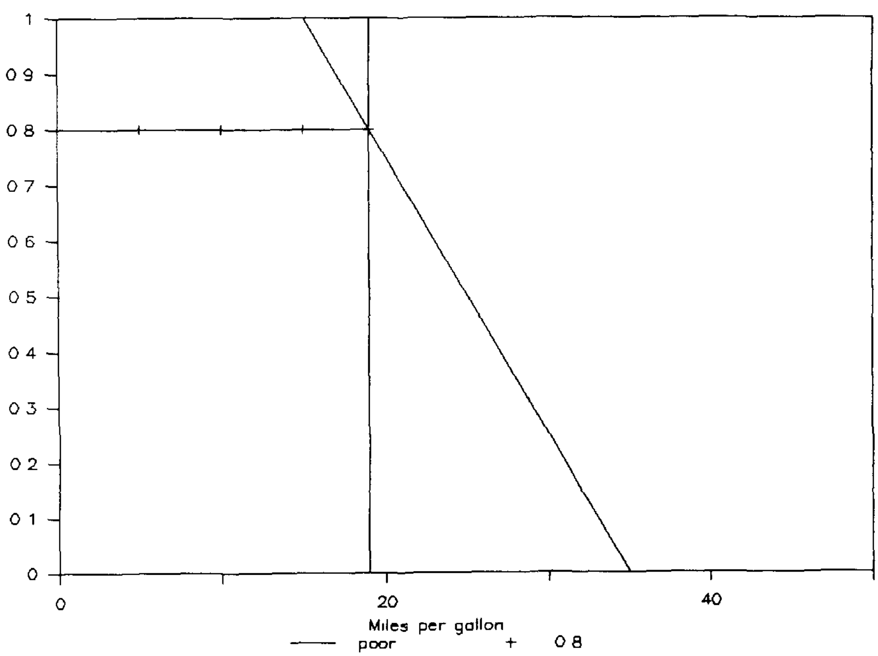

Figure 2. The Fuzzy Set of "Poor" Fuel Economies

Table 3. Truth Values of the Implication Predicate

Lukasiewicz: S1,R1

		Mpg										
		0	5	10	15	20	25	30	35	40	45	50
		1	1	1	1	.75	50	.25	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1
10	0	1	1	1	1	1	1	1	1	1	1	1
20	0	1	1	1	1	1	1	1	1	1	1	1
M 30	0	1	1	1	1	1	1	1	1	1	1	1
p 40	0	1	1	1	1	1	1	1	1	1	1	1
h 50	0	1	1	1	1	1	1	1	1	1	1	1
60	0.80	1	1	1	1	95	.70	.45	.20	.20	20	.20
70	0.94	1	1	1	1	.81	.56	31	06	.06	.06	.06
80	0.97	1	1	1	1	78	.53	.28	.03	03	.03	.03
90	0 98	1	1	1	1	77	.52	27	.02	.02	02	02
100	0 99	1	1	1	1	.76	.51	26	.01	.01	.01	01

Probabilistic: S2

		Mpg										
		0	5	10	15	20	25	30	35	40	45	50
		1	1	1	1	.75	.50	.25	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1
10	0	1	1	1	1	1	1	1	1	1	1	1
20	0	1	1	1	1	1	1	1	1	1	1	1
M 30	0	1	1	1	1	1	1	1	1	1	1	1
p 40	0	1	1	1	1	1	1	1	1	1	1	1
h 50	0	1	1	1	1	1	1	1	1	1	1	1
60	0.80	1	1	1	1	.80	.60	40	.20	.20	20	20
70	0.94	1	1	1	1	.76	.53	.29	.06	.06	.06	06
80	0.97	1	1	1	1	.76	51	27	.03	.03	.03	.03
90	0.98	1	1	1	1	.75	.51	.26	.02	.02	02	02
100	0.99	1	1	1	1	.75	50	26	01	.01	.01	.01

Quotient: R2

		Mpg										
		0	5	10	15	20	25	30	35	40	45	50
		1	1	1	1	.75	.50	.25	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1
10	0	1	1	1	1	1	1	1	1	1	1	1
20	0	1	1	1	1	1	1	1	1	1	1	1
M 30	0	1	1	1	1	1	1	1	1	1	1	1
p 40	0	1	1	1	1	1	1	1	1	1	1	1
h 50	0	1	1	1	1	1	1	1	1	1	1	1
60	0.80	1	1	1	1	.94	.63	.31	0	0	0	0
70	0.94	1	1	1	1	.80	.53	.27	0	0	0	0
80	0.97	1	1	1	1	77	51	.26	0	0	0	0
90	0.98	1	1	1	1	.76	.51	.25	0	0	0	0
100	0 99	1	1	1	1	76	51	.25	0	0	0	0

Kleen-Diens: S3

		Mpg										
		0	5	10	15	20	25	30	35	40	45	50
		1	1	1	1	75	.50	.25	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1
10	0	1	1	1	1	1	1	1	1	1	1	1
20	0	1	1	1	1	1	1	1	1	1	1	1
M 30	0	1	1	1	1	1	1	1	1	1	1	1
p 40	0	1	1	1	1	1	1	1	1	1	1	1
h 50	0	1	1	1	1	1	1	1	1	1	1	1
60	0.80	1	1	1	1	.75	50	25	.20	.20	.20	.20
70	0 94	1	1	1	1	.75	50	25	.06	.06	.06	.06
80	0.97	1	1	1	1	75	.50	.25	.03	.03	.03	03
90	0.98	1	1	1	1	.75	.50	.25	.02	.02	02	.02
100	0.99	1	1	1	1	75	50	.25	.01	.01	01	.01

Brouwer: R3

		Mpg										
		0	5	10	15	20	25	30	35	40	45	50
		1	1	1	1	.75	.50	.25	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1
10	0	1	1	1	1	1	1	1	1	1	1	1
20	0	1	1	1	1	1	1	1	1	1	1	1
M 30	0	1	1	1	1	1	1	1	1	1	1	1
p 40	0	1	1	1	1	1	1	1	1	1	1	1
h 50	0	1	1	1	1	1	1	1	1	1	1	1
60	0 80	1	1	1	1	75	.50	25	0	0	0	0
70	0 94	1	1	1	1	.75	.50	.25	0	0	0	0
80	0.97	1	1	1	1	75	50	25	0	0	0	0
90	0.98	1	1	1	1	.75	.50	.25	0	0	0	0
100	0.99	1	1	1	1	.75	.50	.25	0	0	0	0

A fuzzy knowledge base often is expressed as a collection of arrays, where each array corresponds to an implication formed by two fuzzy predicates, and each entry in an array gives the truth value of that implication for the corresponding pair of elements of the respective universes of discourse. (It is possible to assert that an implication is only partly true, as we did in the case of predicate logic, but the result is equivalent to another implication with a modified consequent that is completely true, so no generality is gained.)

Suppose that "If speed is fast then fuel economy is poor" is one such implication. Table 3 shows the truth values of the implication predicate at

Table 4. Presumption

Lukasiewicz: S1,R1

		Mpg											
		0	5	10	15	20	25	30	35	40	45	50	Pre sump tion
		1	1	1	1	75	.50	.25	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0
M 30	0	0	0	0	0	0	0	0	0	0	0	0	0
p 40	0	0	0	0	0	0	0	0	0	0	0	0	0
h 50	0	0	0	0	0	0	0	0	0	0	0	0	0
60	0.80	0	0	0	0	05	30	.55	.80	.80	80	.80	.80
70	0.94	0	0	0	0	19	44	69	.94	.94	.94	.94	.94
80	0 97	0	0	0	0	22	.47	.72	97	97	97	97	.97
90	0.98	0	0	0	0	.23	48	73	.98	.98	98	.98	.98
100	0.99	0	0	0	0	.24	.49	.74	99	99	99	99	.99

Probabilistic: S2

		Mpg											
		0	5	10	15	20	25	30	35	40	45	50	Pre sump tion
		1	1	1	1	75	.50	.25	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0
M 30	0	0	0	0	0	0	0	0	0	0	0	0	0
p 40	0	0	0	0	0	0	0	0	0	0	0	0	0
h 50	0	0	0	0	0	0	0	0	0	0	0	0	0
60	0.80	0	0	0	0	.20	.40	.60	.80	.80	80	.80	.80
70	0.94	0	0	0	0	.24	.47	.71	.94	.94	.94	.94	.94
80	0.97	0	0	0	0	.24	.49	73	.97	97	.97	.97	97
90	0.98	0	0	0	0	25	.49	.74	98	.98	98	.98	.98
100	0.99	0	0	0	0	.25	.50	.74	.99	99	.99	.99	99

Quotient· R2

		Mpg											
		0	5	10	15	20	25	30	35	40	45	50	Pre sump tion
		1	1	1	1	.75	50	.25	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0
M 30	0	0	0	0	0	0	0	0	0	0	0	0	0
p 40	0	0	0	0	0	0	0	0	0	0	0	0	0
h 50	0	0	0	0	0	0	0	0	0	0	0	0	0
60	0.80	0	0	0	0	0+	0+	0+	0	0	0	0	0+
70	0.94	0	0	0	0	0+	0+	0+	0	0	0	0	0+
80	0 97	0	0	0	0	0+	0+	0+	0	0	0	0	0+
90	0.98	0	0	0	0	0+	0+	0+	0	0	0	0	0+
100	0.99	0	0	0	0	0+	0+	0+	0	0	0	0	0+

Kleen-Diens. S3

		Mpg											
		0	5	10	15	20	25	30	35	40	45	50	Pre sump tion
		1	1	1	1	.75	.50	.25	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0
M 30	0	0	0	0	0	0	0	0	0	0	0	0	0
p 40	0	0	0	0	0	0	0	0	0	0	0	0	0
h 50	0	0	0	0	0	0	0	0	0	0	0	0	0
60	0.80	0	0	0	0	.25	50	.75	80	.80	.80	.80	.80
70	0.94	0	0	0	0	.25	.50	.75	.94	.94	.94	.94	.94
80	0.97	0	0	0	0	25	.50	.75	.97	.97	.97	97	.97
90	0.98	0	0	0	0	25	50	.75	98	.98	98	.98	98
100	0 99	0	0	0	0	25	50	.75	99	.99	99	.99	.99

Brouwer: R3

		Mpg											
		0	5	10	15	20	25	30	35	40	45	50	Pre sump tion
		1	1	1	1	75	.50	25	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0
M 30	0	0	0	0	0	0	0	0	0	0	0	0	0
p 40	0	0	0	0	0	0	0	0	0	0	0	0	0
h 50	0	0	0	0	0	0	0	0	0	0	0	0	0
60	0.80	0	0	0	0	.75	.50	25	0	0	0	0	.75
70	0.94	0	0	0	0	75	50	.25	0	0	0	0	75
80	0.97	0	0	0	0	75	50	25	0	0	0	0	75
90	0.98	0	0	0	0	75	.50	.25	0	0	0	0	75
100	0.99	0	0	0	0	75	50	25	0	0	0	0	75

selected (x_i, y_j) pairs, where x_i is a particular rate of speed measured in miles per hour and y_j is a particular level of fuel economy measured in miles per gallon, for each of the five logics discussed above Table 4 shows the lower bounds on the truth of the speed component of the pair, defined by the lowest antecedent truth value of x_i that could generate the given truth value of the implication within the logic in question (presumption). The greatest lower bound in each row gives the overall presumption derived from the rule, a logical

Table 5. Prejudice

Lukasiewicz: S1,R1

		Mpg										
		0	5	10	15	20	25	30	35	40	45	50
		1	1	1	1	.75	.50	.25	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1
10	0	1	1	1	1	1	1	1	1	1	1	1
20	0	1	1	1	1	1	1	1	1	1	1	1
M 30	0	1	1	1	1	1	1	1	1	1	1	1
p 40	0	1	1	1	1	1	1	1	1	1	1	1
h 50	0	1	1	1	1	1	1	1	1	1	1	1
60	0.80	1	1	1	1	.95	.70	.45	20	.20	.20	20
70	0.94	1	1	1	1	.81	.56	31	06	.06	.06	.06
80	0.97	1	1	1	1	.78	53	28	.03	.03	.03	.03
90	0.98	1	1	1	1	77	52	.27	.02	.02	.02	.02
100	0.99	1	1	1	1	76	51	.26	.01	.01	.01	.01
Prejudice		1	1	1	1	76	.51	26	01	01	.01	.01

Probabilistic: S2

		Mpg										
		0	5	10	15	20	25	30	35	40	45	50
		1	1	1	1	.75	.50	.25	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1
10	0	1	1	1	1	1	1	1	1	1	1	1
20	0	1	1	1	1	1	1	1	1	1	1	1
M 30	0	1	1	1	1	1	1	1	1	1	1	1
p 40	0	1	1	1	1	1	1	1	1	1	1	1
h 50	0	1	1	1	1	1	1	1	1	1	1	1
60	0.80	1	1	1	1	.80	.60	.40	.20	.20	.20	.20
70	0 94	1	1	1	1	.76	.53	.29	.06	.06	.06	.06
80	0.97	1	1	1	1	.76	.51	.27	.03	.03	.03	.03
90	0.98	1	1	1	1	.75	.51	.26	.02	.02	.02	.02
100	0.99	1	1	1	1	.75	.50	.26	.01	.01	.01	.01
Prejudice		1	1	1	1	.75	.50	.26	.01	.01	.01	.01

Quotient· R2

		Mpg										
		0	5	10	15	20	25	30	35	40	45	50
		1	1	1	1	.75	.50	.25	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1
10	0	1	1	1	1	1	1	1	1	1	1	1
20	0	1	1	1	1	1	1	1	1	1	1	1
M 30	0	1	1	1	1	1	1	1	1	1	1	1
p 40	0	1	1	1	1	1	1	1	1	1	1	1
h 50	0	1	1	1	1	1	1	1	1	1	1	1
60	0.80	1	1	1	1	.94	.63	.31	0	0	0	0
70	0.94	1	1	1	1	.80	.53	.27	0	0	0	0
80	0.97	1	1	1	1	.77	.51	.26	0	0	0	0
90	0.98	1	1	1	1	.76	.51	.25	0	0	0	0
100	0.99	1	1	1	1	.76	.51	.25	0	0	0	0
Prejudice		1	1	1	1	.76	.51	.25	0	0	0	0

Kleen-Diens: S3

		Mpg										
		0	5	10	15	20	25	30	35	40	45	50
		1	1	1	1	.75	.50	.25	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1
10	0	1	1	1	1	1	1	1	1	1	1	1
20	0	1	1	1	1	1	1	1	1	1	1	1
M 30	0	1	1	1	1	1	1	1	1	1	1	1
p 40	0	1	1	1	1	1	1	1	1	1	1	1
h 50	0	1	1	1	1	1	1	1	1	1	1	1
60	0.80	1	1	1	1	.75	.50	.25	.20	.20	.20	.20
70	0.94	1	1	1	1	.75	.50	.25	.06	.06	.06	.06
80	0.97	1	1	1	1	.75	.50	.25	.03	.03	.03	03
90	0.98	1	1	1	1	.75	.50	.25	.02	.02	.02	.02
100	0.99	1	1	1	1	.75	.50	.25	.01	.01	.01	.01
Prejudice		1	1	1	1	.75	.50	.25	.01	.01	.01	01

Brouwer: R3

		Mpg										
		0	5	10	15	20	25	30	35	40	45	50
		1	1	1	1	.75	.50	.25	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1
10	0	1	1	1	1	1	1	1	1	1	1	1
20	0	1	1	1	1	1	1	1	1	1	1	1
M 30	0	1	1	1	1	1	1	1	1	1	1	1
p 40	0	1	1	1	1	1	1	1	1	1	1	1
h 50	0	1	1	1	1	1	1	1	1	1	1	1
60	0 80	1	1	1	1	.75	.50	.25	0	0	0	0
70	0.94	1	1	1	1	.75	.50	.25	0	0	0	0
80	0.97	1	1	1	1	.75	.50	.25	0	0	0	0
90	0.98	1	1	1	1	75	.50	25	0	0	0	0
100	0.99	1	1	1	1	75	.50	.25	0	0	0	0
Prejudice		1	1	1	1	.75	.50	.25	0	0	0	0

inconsistency occurs if a predicate of speed is asserted that assigns any particular rate of speed a membership grade less than the membership grade assigned to it by presumption. An example of such a predicate is the datum "You drive very fast." Table 5 shows a similar analysis for prejudice; the least upper bound for each column gives the overall prejudice.

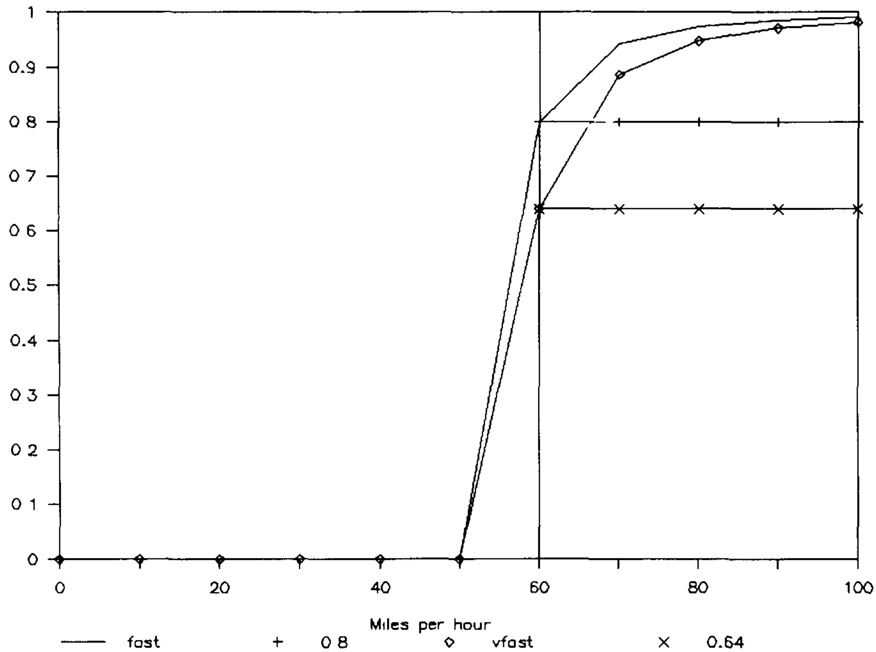

Figure 3. "Fast" Speeds and "Very Fast" Speeds

Set Logic

Consider the statement "People who drive fast have poor fuel economy," which asserts that each person belongs to the fuzzy set of people with poor fuel economy at least as much as he or she belongs to the fuzzy set of people who drive fast. To understand this relation, consider the fuzzy set of "fast" speeds and the fuzzy set of "poor" fuel economies shown in Figures 1 and 2 Suppose we want to deduce the fuel economy of a person whose driving speed is 60 mph. Figure 1 displays a crisp value for a speed of 60 mph superimposed upon the membership function of "fast" in the speed universe of discourse. Persons whose speed is 60 belong with degree 0.8 to the set of people who drive fast, since this is the degree of membership of their speed in the set of "fast" speeds

Figure 2 portrays the membership function of "poor" in the fuel economy universe of discourse. The persons in the set whose fuel economies are at least 0 8 compatible with "poor" are shown in Figure 2 to have fuel economies in the crisp interval [0, 19] mpg Thus, we infer that no one who habitually drives 60 mph has a fuel economy better than 19 mpg.

Now suppose that the datum regarding person P's driving habits is "Person P drives very fast." Figure 3 graphs the membership functions of "fast" (the antecedent) and "very fast" (the datum). For any particular speed, say 60 mph, we can find a membership grade in person P's speed, in this case 0.64, and a

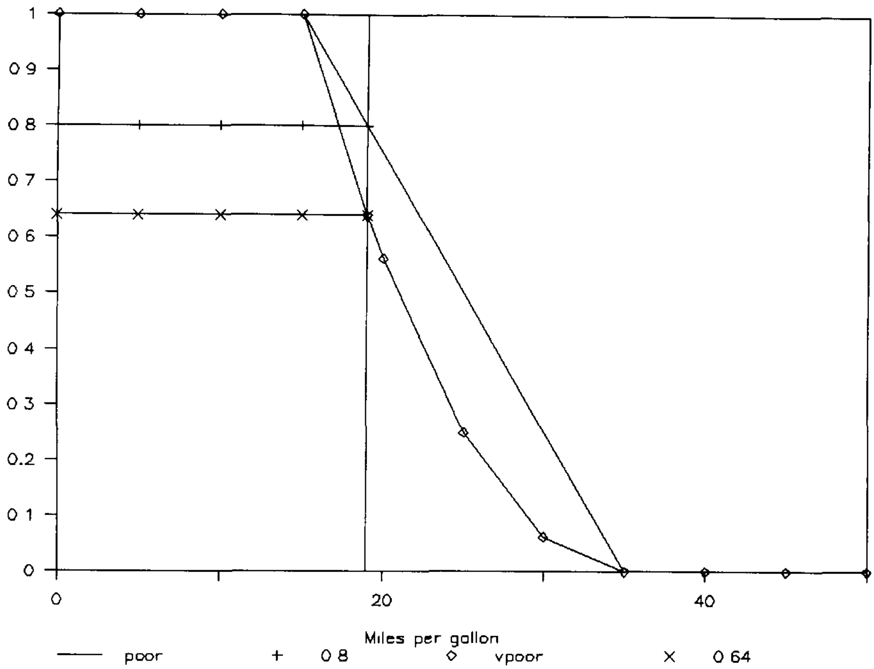

Figure 4. "Poor" Fuel Economies and "Very Poor" Fuel Economies

generally different membership grade in the antecedent, in this case 0 8 From this information, which pertains only to 60 mph, we can derive one level set of the fuzzy set of mpg ratings that makes up person P's inferred fuel economy.

We begin by finding the level set of the consequent "poor" corresponding to 0.8, the membership grade of 60 mph in the antecedent "fast." This level set is the interval from 0 to 19 mpg If 60 mph were perfectly compatible with person P's driving, as it was in the previous example, this interval would be perfectly compatible with person P's fuel economy. However, since we now assume that person P's driving is not "fast" but "very fast," 60 mph is only 0.64 compatible with person P's driving, so we infer that the corresponding 0 8 level set of the consequent "poor" is only 0 64 compatible with person P's fuel economy

Thus, the upper horizontal line in Figure 4 representing the 0 8 level set of the consequent "poor" is moved down to a membership grade of 0.64; the lower horizontal line represents the 0.64 level set in the inferred fuel economy for person P.

We repeat this process for all possible miles per hour speeds. That is, we find the level set of "poor" corresponding to the membership grade of each particular speed x_i mph in the antecedent "fast," and assign this level set a membership in P's inferred fuel economy equal to the membership grade of the speed x_i in the datum "P drives very fast." These level sets, defined in the fuel economy (mpg) universe of discourse, collectively trace out the lower curve in Figure 4. Generating the inferred fuel economy by applying the datum "P drives very fast" to the rule "People who drive fast have poor fuel economy" thus allows us to infer that P has very poor fuel economy.

CONCLUSION

Confirmation and denial are two recently introduced modes of inference in fuzzy logic (Bandler and Kohout [3], Hall [4], Schwartz [5]). We have examined these modes together with the standard modes of modus ponens and modus tollens in the contexts of propositional logic and predicate logic and have derived the closely related inference modes of presumption and prejudice. In the process, we have uncovered some fundamental difficulties in how fuzzy implication is to be interpreted in a fuzzy expert system given the a priori restrictions placed on the model of the application by presumption and prejudice, and to a lesser extent by confirmation and denial.

One way of escaping these problems is to use a logic based on fuzzy sets rather than on multivalent truth values, as discussed in the section on set logic The principal drawback of this approach is some loss of expressive flexibility, in effect, the system of fuzzy set logic described here limits the choice of implication operator to a single, rather nonfuzzy choice. the "standard strict" implication operator. In future research, we intend to relax this restriction somewhat through the use of aspects of Zadeh's concept of "usuality" [19]

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